Karlsruhe Institute of Technology

# Text Indexing

## Lecture 02: Suffix Trees and Suffix Arrays

Florian Kurpicz



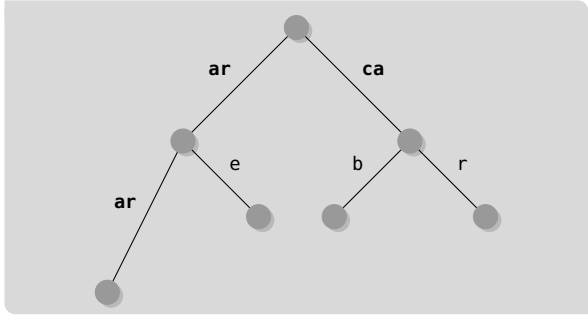
# PINGO

https://pingo.scc.kit.edu/289240

# Recap: Compact Trie

## Definition: Compact Trie

- A compact trie is a trie where all branchless paths are replaced by a single edge.
- The label of the new edge is the concatenation of the replaced edges' labels.




## Next

A full-text index for a text $T$ is

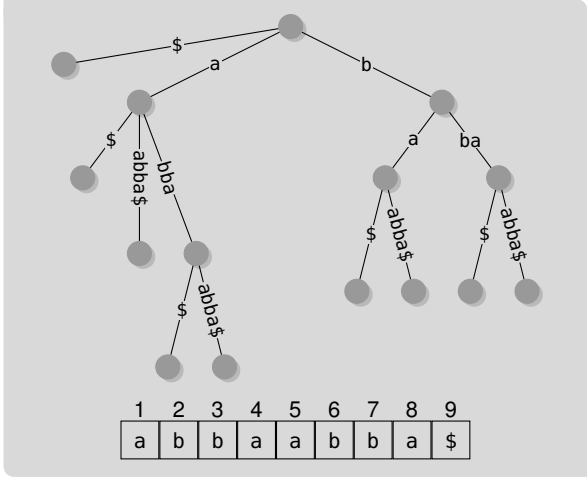
- a data structure that
- allows to answer queries on $T$ faster than naive
- we are interested in *pattern matching* queries
- how to use tries to create full-text index

# Suffix Tree (1/4)

## Definition: Suffix Tree [Wei73]

A suffix tree ($ST$) for a text $T$ of length $n$ is a

- compact trie
- over $S = \{T[1..n], T[2..n], \ldots, T[n..n]\}$
  - suffixes are prefix-free due to sentinel



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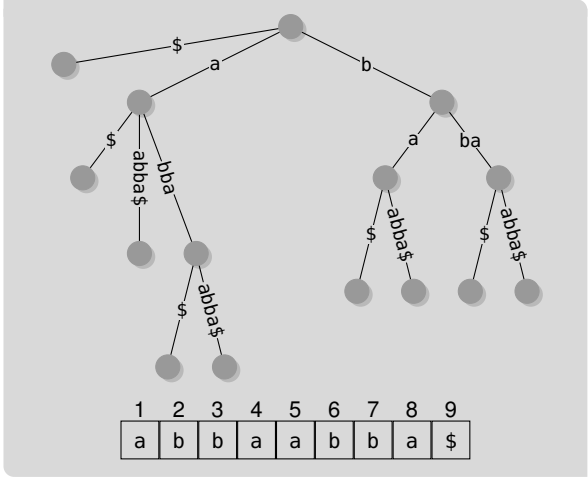
Let $G = (V, E)$ be a compact trie with root $r$ and a node $v \in V$, then

- $\lambda(v)$ is the concatenation of labels from $r$ to $v$
- $d(v) = |\lambda(v)|$ is the string-depth of $v$
  - string depth $\neq$ depth



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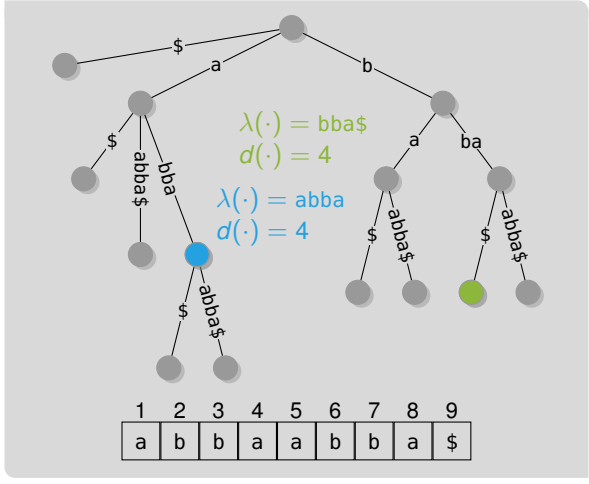


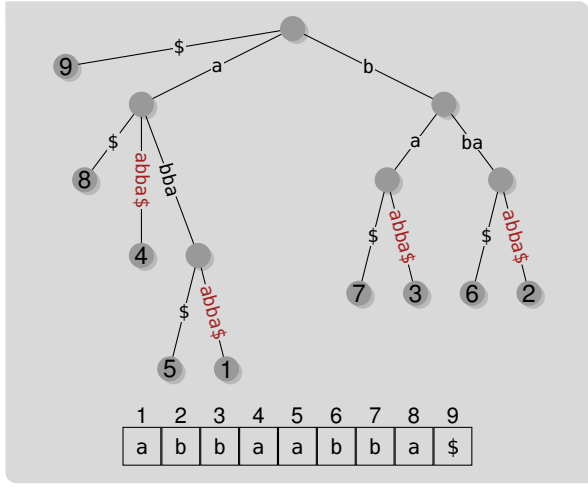
# Suffix Tree (2/4)

## Representing Labels

- explicit edge labels require $O(n^2)$ words space

## Suffix Information

- label leaves with corresponding suffix
  - will be important later on



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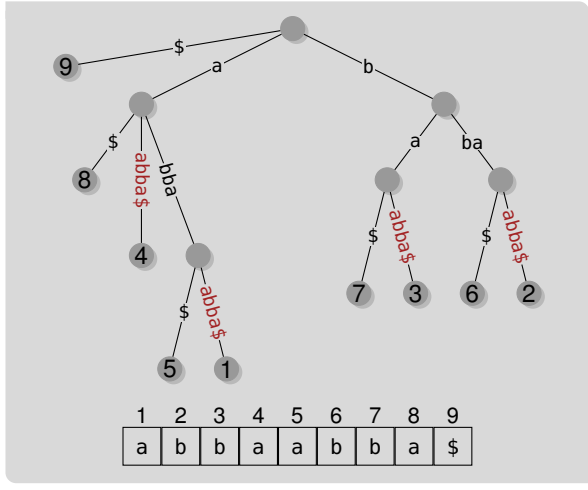

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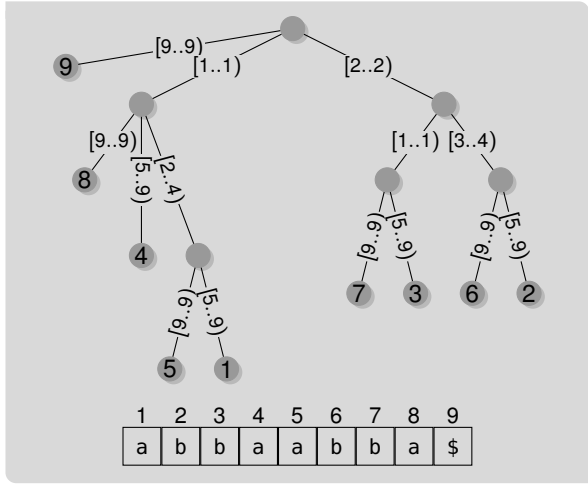

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- for simplicity, we show text

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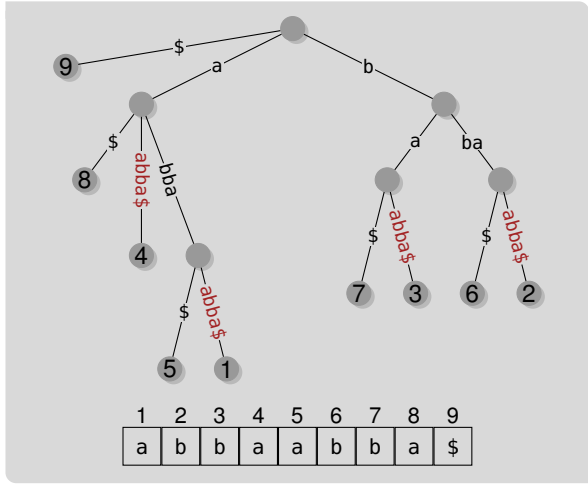
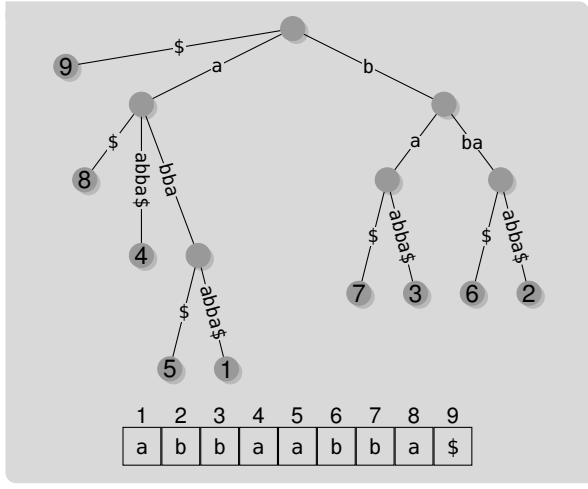


# Suffix Tree (3/4)

## Pattern Matching using Suffix Trees

- Pattern $P[1..m]$
- start at the root and follow edges
- query time depends on representation of children

- $O(m)$ time using $O(n\sigma)$ words space



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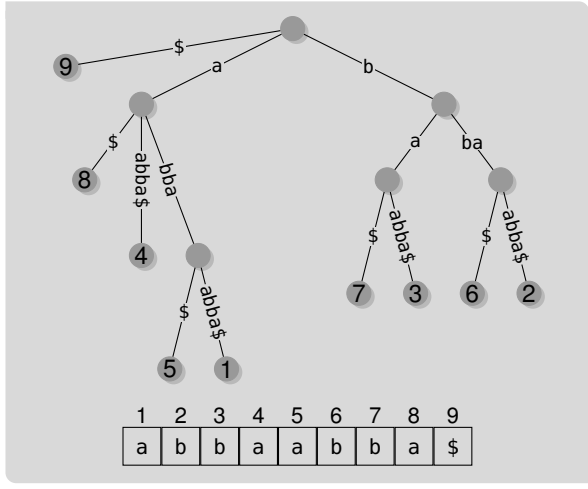
- $O(m \cdot \lg \sigma)$ time with $O(n)$ words space



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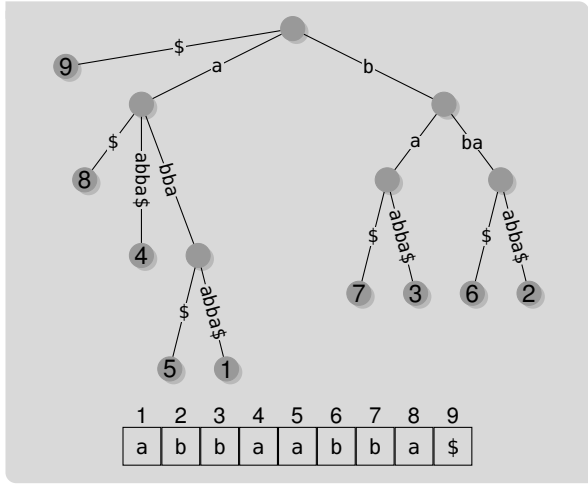
- $O(m + \lg \sigma)$ time with $O(n)$ words space



# Suffix Tree (4/4)

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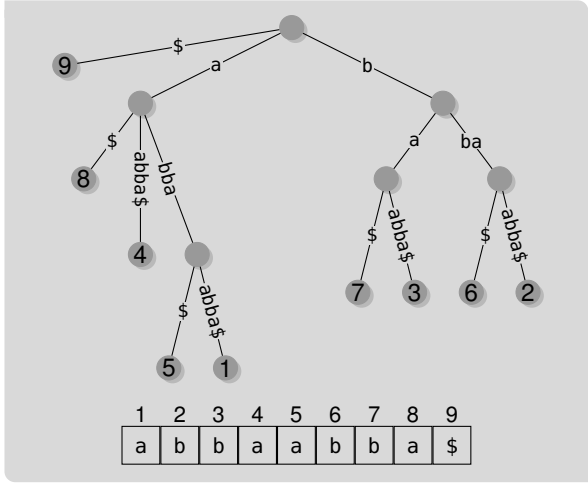

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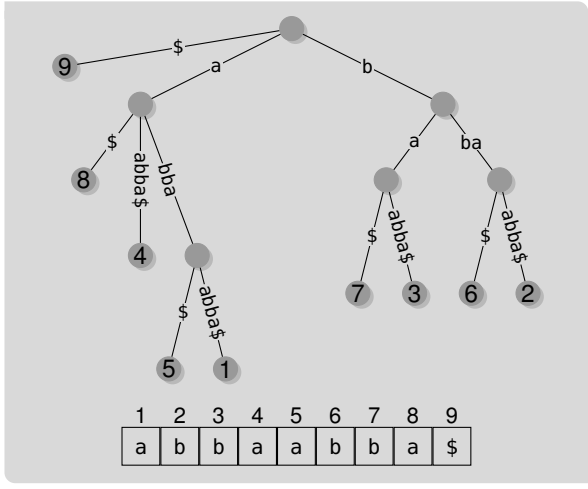

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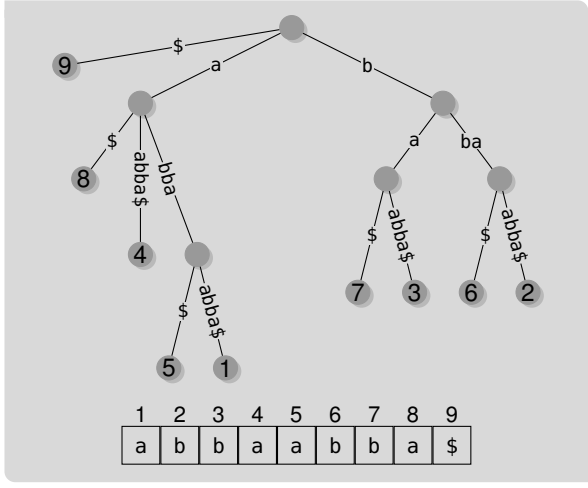

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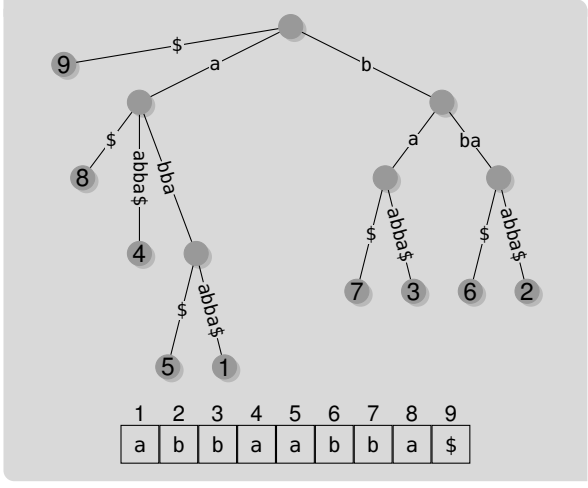

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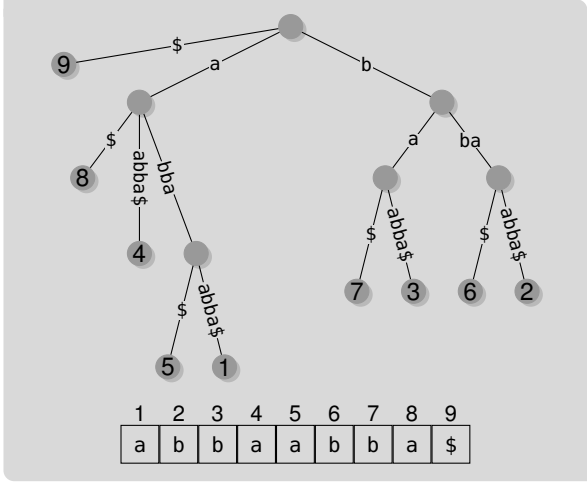

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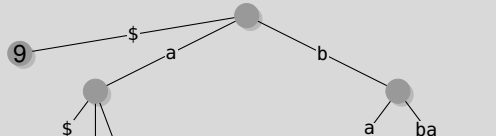

next, suffix array construction

# Suffix Array and LCP-Array

## Definition: Suffix Array [GBS92; MM93]

Given a text $T$ of length $n$, the **suffix array** (SA) is a permutation of $[1..n]$, such that for $i \leq j \in [1..n]$

$$T[SA[i]..n] \leq T[SA[j]..n]$$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | a | b | a | b | c | a | b | c | a | b | b | a | $ |
| $SA$ | 13 | 12 | 1 | 9 | 6 | 3 | 11 | 2 | 10 | 7 | 4 | 8 | 5 |
| $LCP$ | 0 | 0 | 1 | 2 | 2 | 5 | 0 | 2 | 1 | 1 | 4 | 0 | 3 |

$
a$
abaabcabcabba$
abba$
abcabba$
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ba$
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## Definition: Longest Common Prefix Array

Given a text $T$ of length $n$ and its SA, the **LCP-array** is defined as

$$LCP[i] = \begin{cases} 0 & i = 1 \\ \max\{\ell \colon T[SA[i]..SA[i]+\ell) = \\ \qquad T[SA[i-1]..SA[i-1]+\ell)\} & i \neq 1 \end{cases}$$

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| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
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# Pattern Matching with the Suffix Array (1/2)

**Function** SeachSA($T, SA[1..n], P[1..m]$)**:**

1 $\ell = 1, r = n + 1$
2 **while** $\ell < r$ **do**
3 $\quad i = \lfloor (\ell + r)/2 \rfloor$
4 $\quad$ **if** $P > T[SA[i]..SA[i] + m)$ **then**
5 $\quad\quad \ell = i + 1$
6 $\quad$ **else** $r = i$
7 $s = \ell, \ell = \ell - 1, r = n$
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- pattern $P =$ abc

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**Lemma: Running Time SeachSA**

The SeachSA answers counting queries in $O(m \lg n)$ time and reporting queries in $O(m \lg n + occ)$ time

**Proof (Sketch)**

- two binary searches on the $SA$ in $O(lgn)$ time

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- each comparison requires $O(m)$ time
  - counting in $O(1)$ additional time

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6  | s = ℓ, ℓ = ℓ − 1, r = n
7  | while ℓ < r do
8  |   | i = ⌈ℓ + r/2⌉
9  |   | if P = T[SA[i]..SA[i] + m) then ℓ = i
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11 | return [s, r]
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## Lemma: Running Time SeachSA

The SeachSA answers counting queries in $O(m \lg n)$ time and reporting queries in $O(m \lg n + occ)$ time

## Proof (Sketch)

- two binary searches on the $SA$ in $O(lgn)$ time
- each comparison requires $O(m)$ time
- counting in $O(1)$ additional time

# Pattern Matching with the Suffix Array (2/2)

**Function** SeachSA($T$, $SA[1..n]$, $P[1..m]$)**:**

```
1  | ℓ = 1, r = n + 1
2  | while ℓ < r do
3  |   | i = ⌊(ℓ + r)/2⌋
4  |   | if P > T[SA[i]..SA[i] + m) then
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5  |   | else r = i
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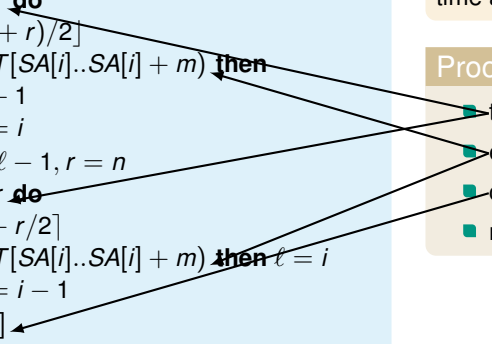

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**Function** SeachSA($T$, $SA[1..n]$, $P[1..m]$)**:**

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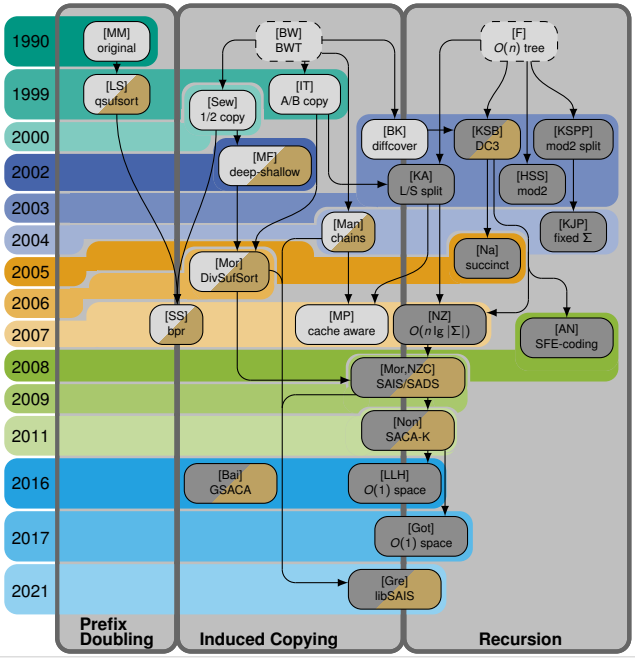
# Preview: Improving Running Time with LCP-Array

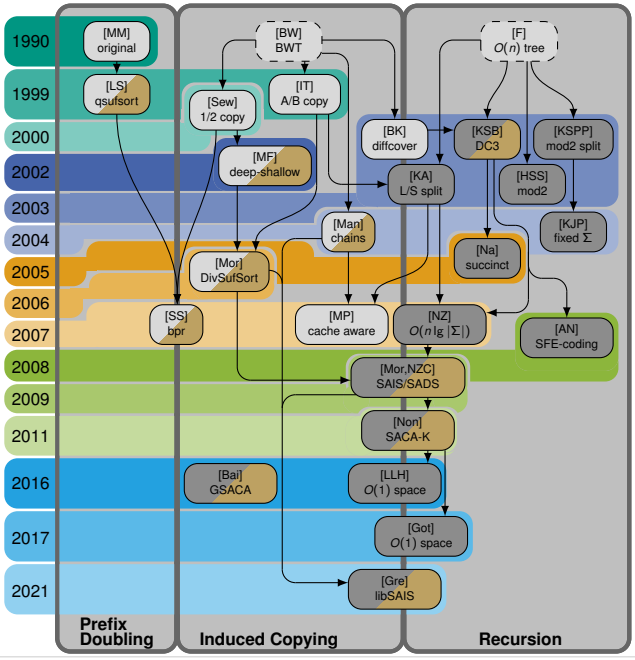
- next lecture: $O(m + \lg n)$ and $O(m + \lg n + occ)$ time
- requires additional indices on LCP-array

- now: how to compute the suffix array directly ℹ without the suffix tree




## Timeline Sequential Suffix Sorting

- based on [Bah+19; Bin18; Kur20; PST07]
- **darker grey**: linear running time
- **brown**: available implementation

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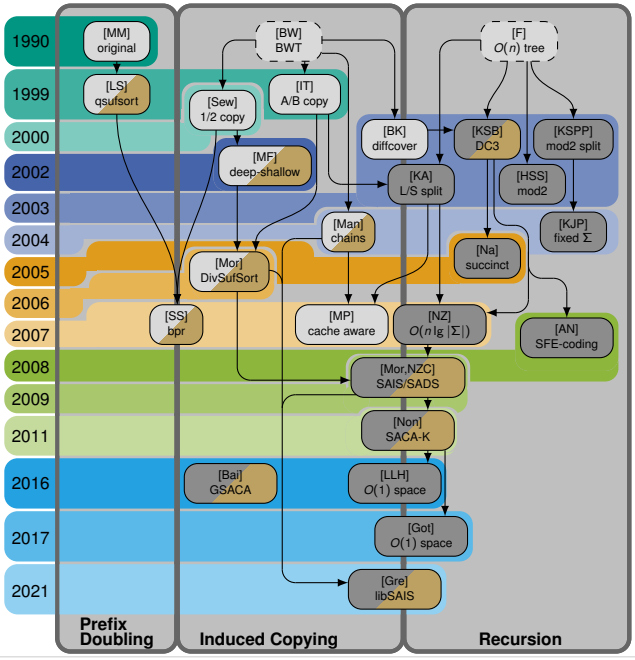
## Special Mentions

- DC3 first $O(n)$ algorithm
- $O(n)$ running time and $O(1)$ space for integer alphabets possible

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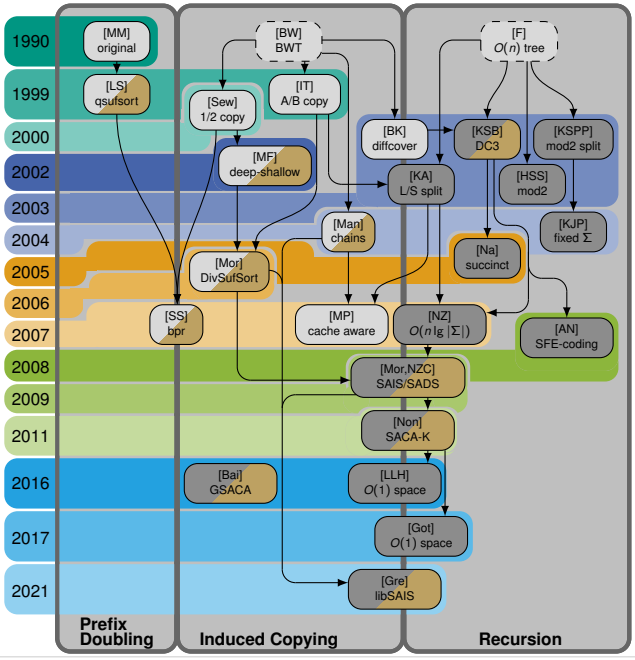


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- since 2021: libSAIS fastest in practice with $O(n)$ running time

# Suffix Array Induced Sorting: Overview

## The Idea: Inducing

Given a text $T$ of length $n$ and two positions $i, j \in [1..n]$ with $T[i] = T[j]$, then

$$T[i..n] < T[j..n] \iff T[i+1..n] < T[j+1..n]$$

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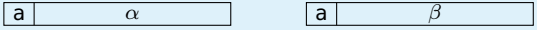

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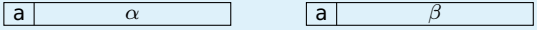


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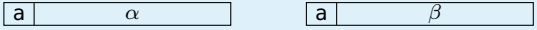


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- classification
- sort special substrings/suffixes recursively
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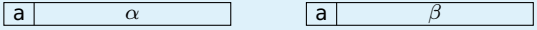


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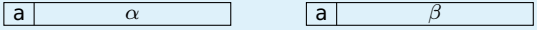


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## Roadmap

- classification
- inducing
- sorting special suffixes

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## Definition: Type $L/S$ Suffixes

Given a text $T$ of length $n$ and $i \in [1..n]$, then

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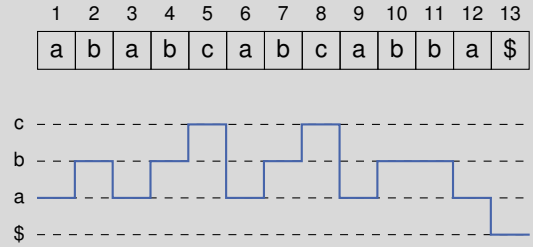

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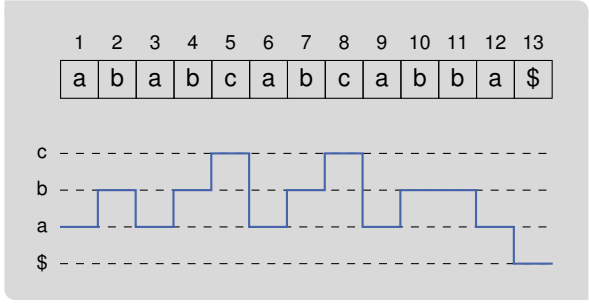

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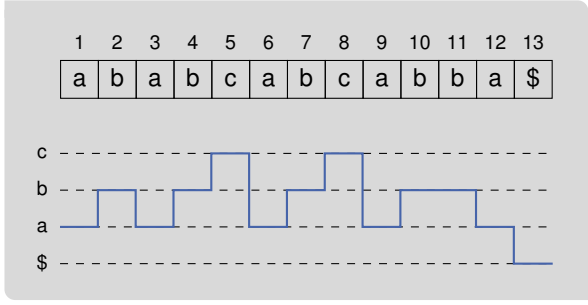

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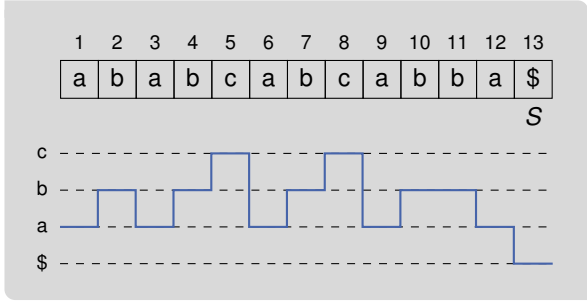

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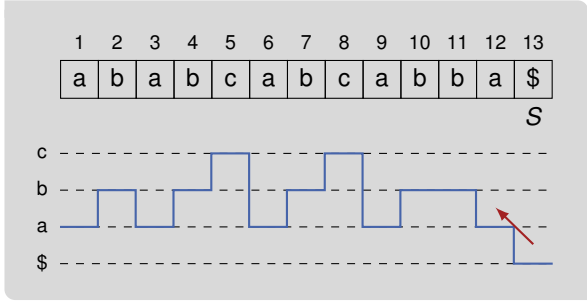

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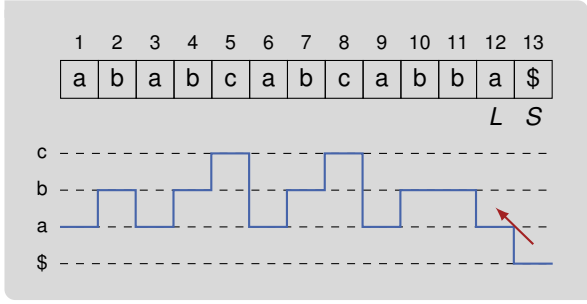

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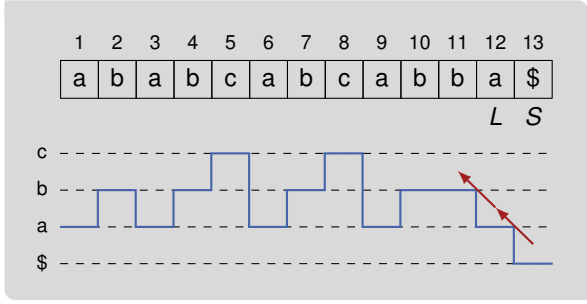

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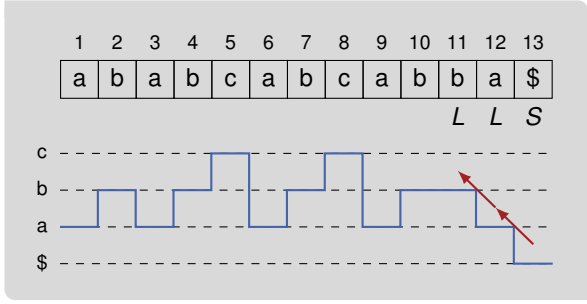

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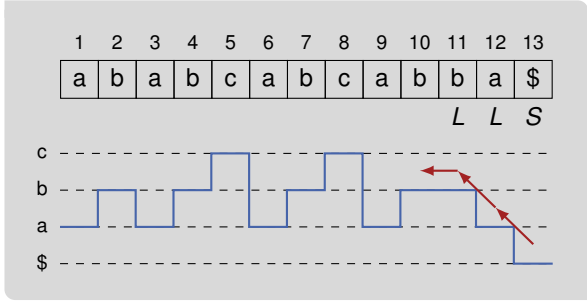

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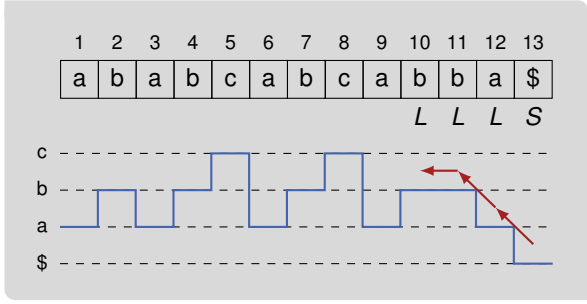

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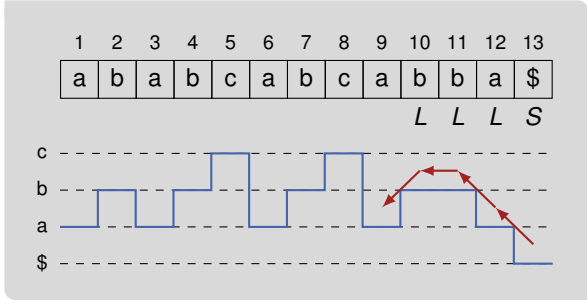

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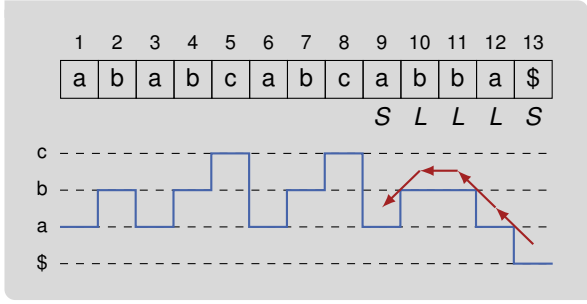

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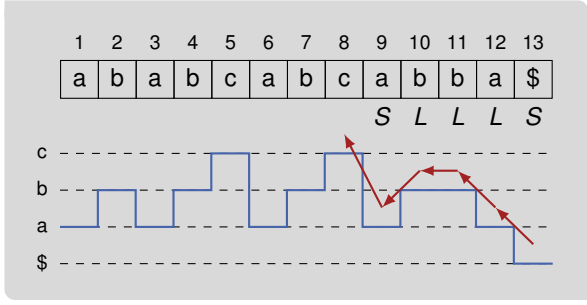

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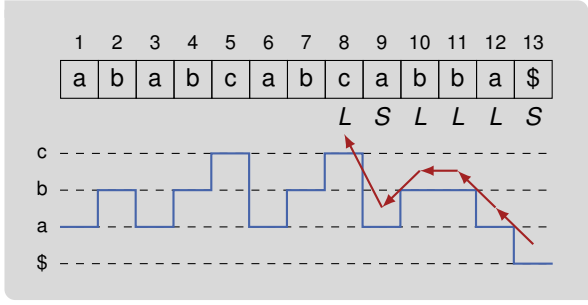

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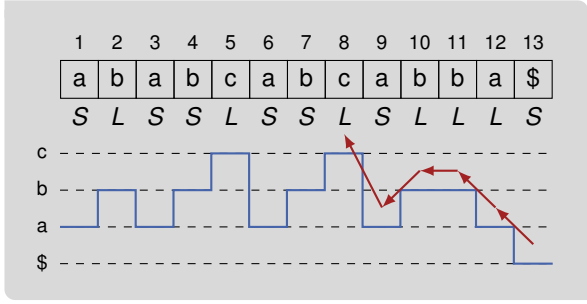

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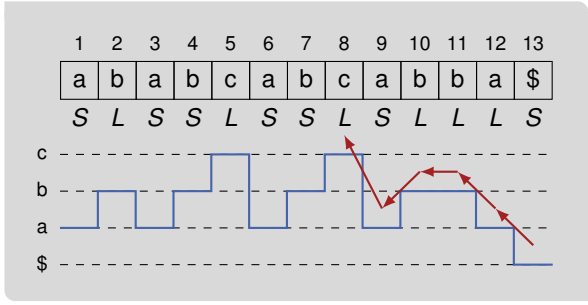
## Definition: Leftmost $S$ Suffixes

Given a text $T$ of length $n$, $i \in [2..n]$ such that $T[i..n]$ has type $S$ and $T[i-1..n]$ has type $L$, then $T[i..n]$ is called **leftmost $S$ suffix (LMS)**.

- denoted by $S^\star$



# Suffix Array Induced Sorting: Classification (1/2)

## Definition: Type $L/S$ Suffixes

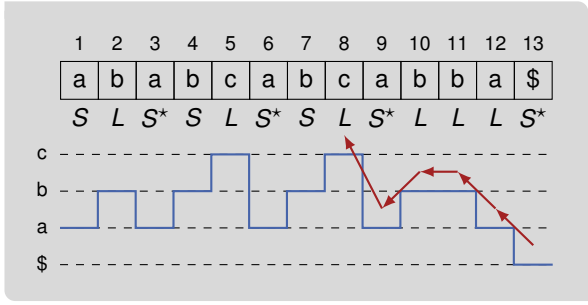
Given a text $T$ of length $n$ and $i \in [1..n]$, then

- $T[i] < T[i+1]$ or $i = n \Rightarrow T[i..n]$ has **type $S$**
- $T[i] > T[i+1] \Rightarrow T[i..n]$ has **type $L$**
- $T[i] = T[i+1] \Rightarrow T[i..n]$ has $T[i+1..n]$'s type.

## Definition: Leftmost $S$ Suffixes

Given a text $T$ of length $n$, $i \in [2..n]$ such that $T[i..n]$ has type $S$ and $T[i-1..n]$ has type $L$, then $T[i..n]$ is called **leftmost $S$ suffix (LMS)**.

- denoted by $S^\star$



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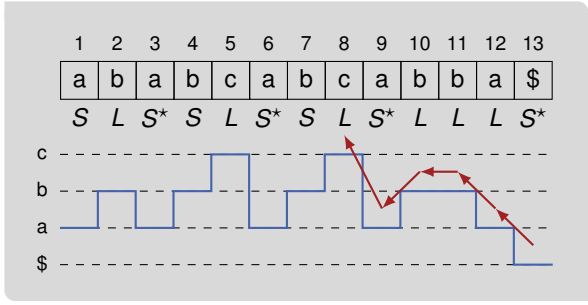
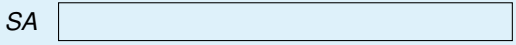



- scan text from right to left
- do not store types explicitly ℹ initially, we are only interested in LMS-suffixes

# Suffix Array Induced Sorting: Classification (2/2)

- partition suffix array based text's histogram



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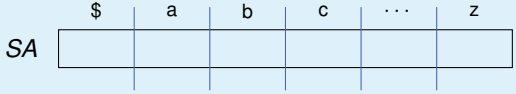

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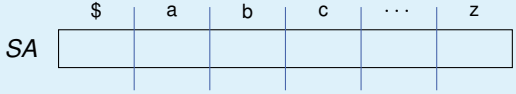

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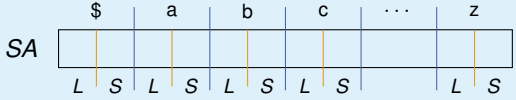

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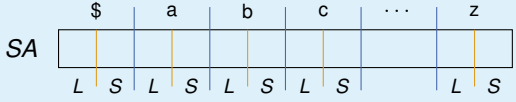


## Lemma: Order of $L/S$ Suffixes

Given a text $T$ of length $n$, a type $L$ suffixes $T[i..n]$ and a type $S$ $T[j..n]$ with $\alpha = T[i] = T[i]$, then

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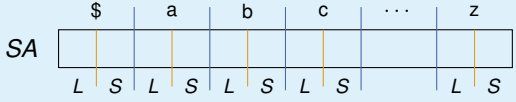


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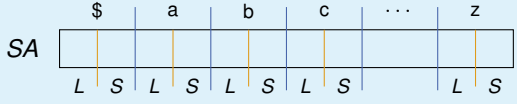


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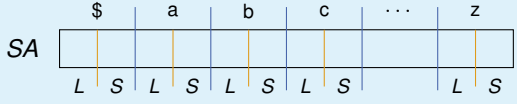


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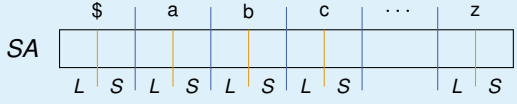


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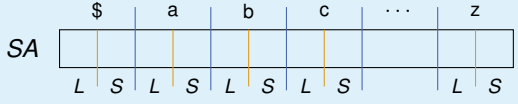


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# Suffix Array Induced Sorting: Inducing (1/2)

## Lemma: Inducing

If $T[i+1..n] < T[j+1..n]$ and $T[i] = T[j]$ then

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| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | a | b | a | b | c | a | b | c | a | b | b | a | $ |
| $SA$ | 13 | 12 | 1 | 9 | 6 | 3 | 11 | 2 | 10 | 7 | 4 | 8 | 5 |
| | $ | a $ | a b a b c a b c a b b a $ | a b b a $ | a b c a b b a $ | a b c a b c a b b a $ | b a $ | b a b c a b c a b b a $ | b b a $ | b c a b b a $ | b c a b c a b b a $ | c a b b a $ | c a b c a b b a $ |

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## Proof (Sketch)

- similar to order of $L/S$ suffixes
- there is a leftmost character where $T[i+1..n]$ and $T[j+1..n]$ differ
- $T[i..n]$ and $T[j..n]$ differ at the same character

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- Initialization
  - initialize each entry in SA with "—"
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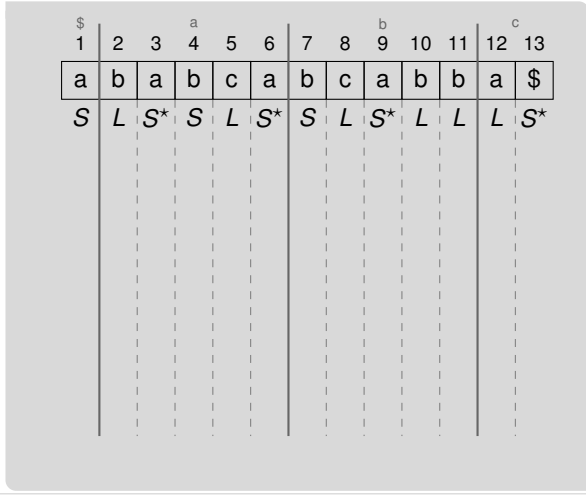

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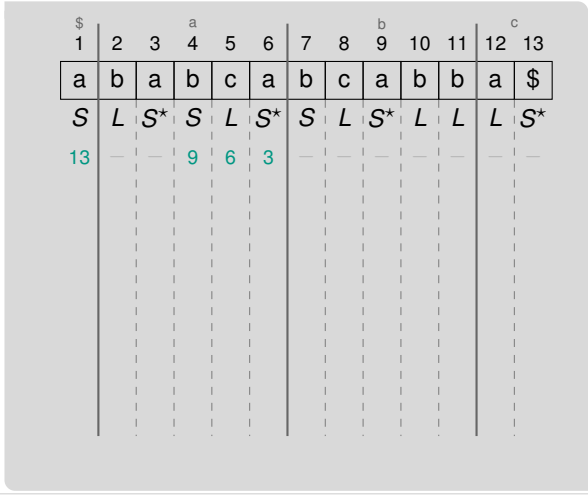

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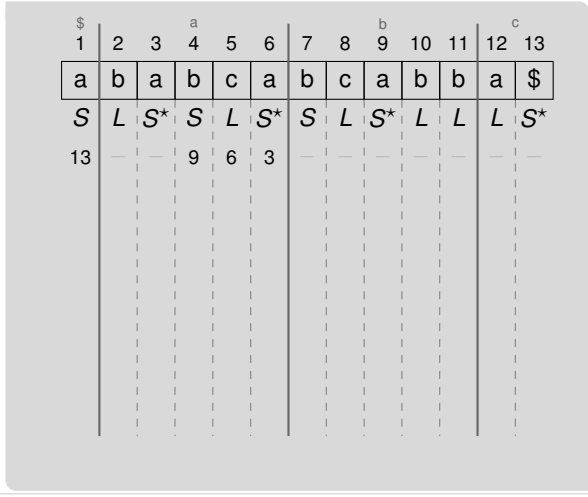

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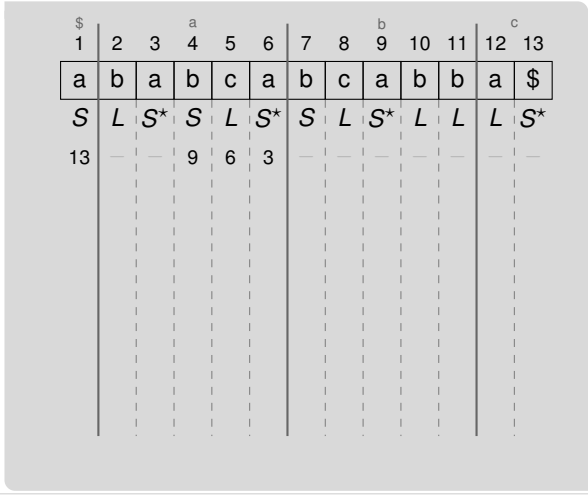

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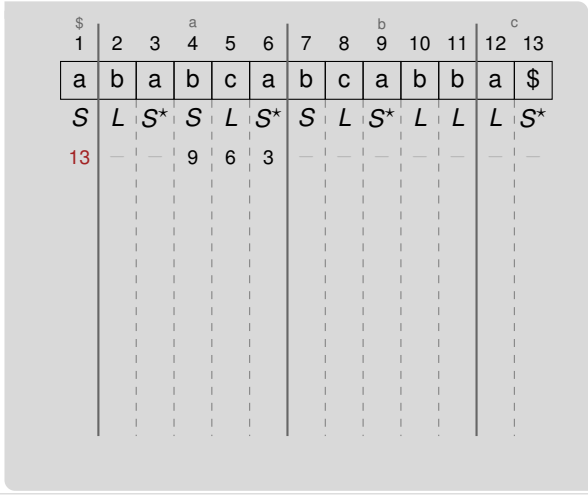

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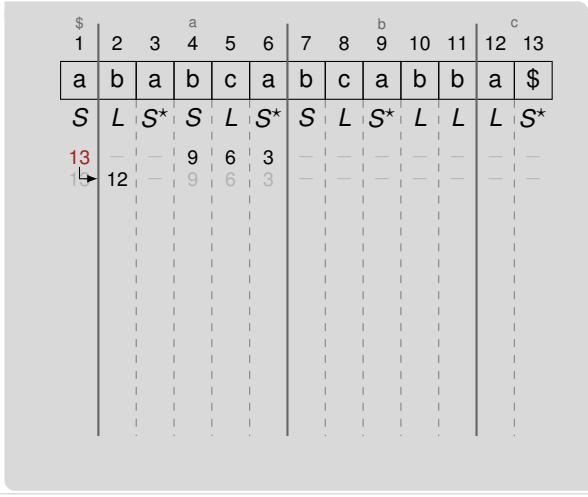

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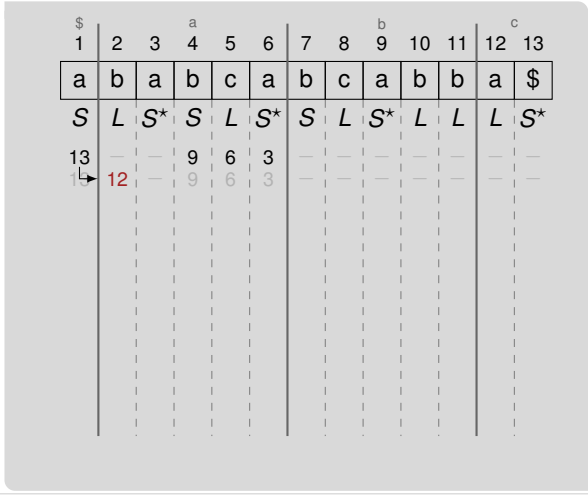

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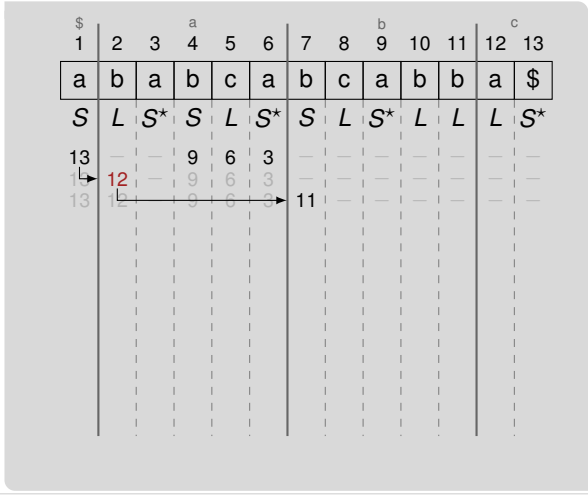

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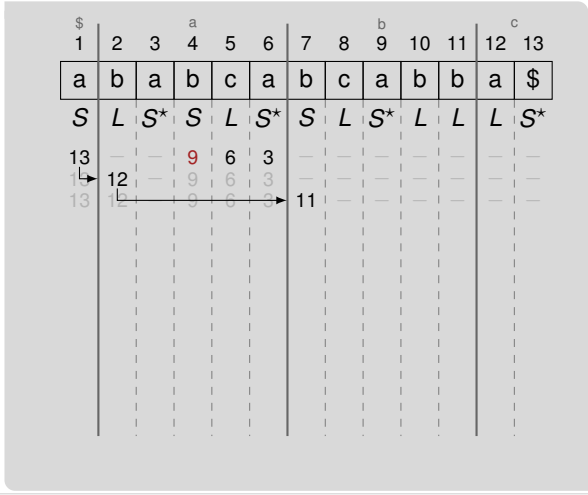

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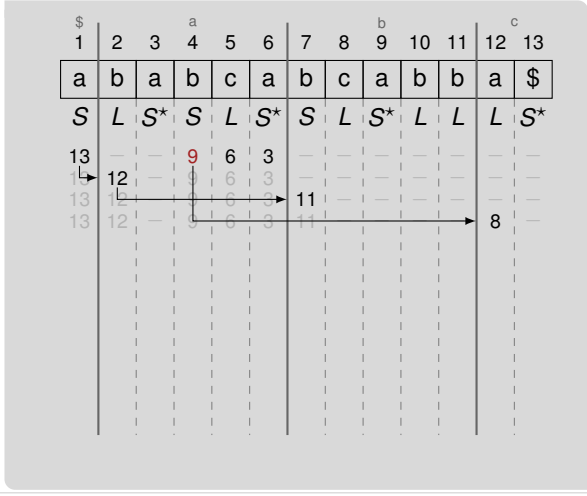

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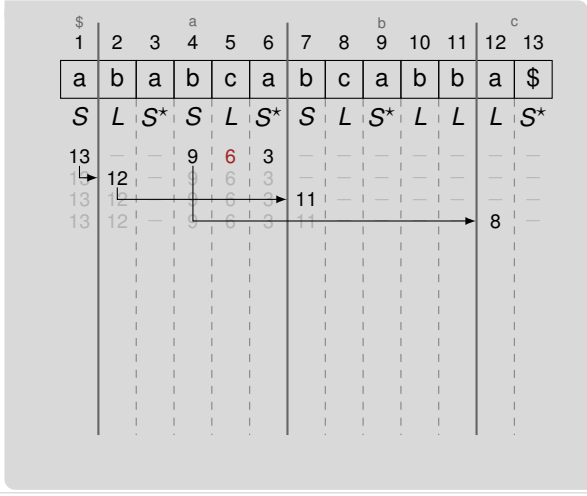

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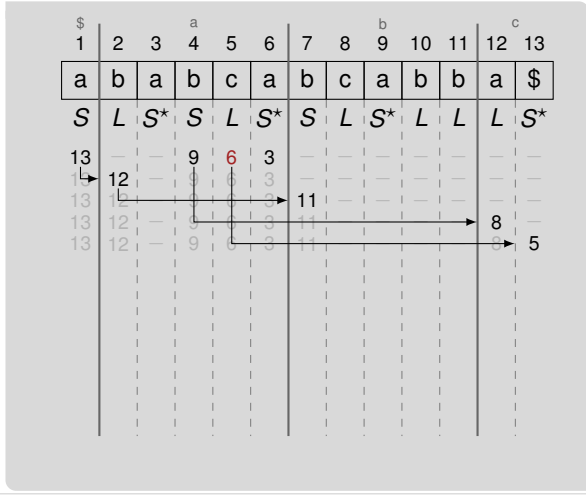

# Suffix Array Induced Sorting: Inducing (2/2)

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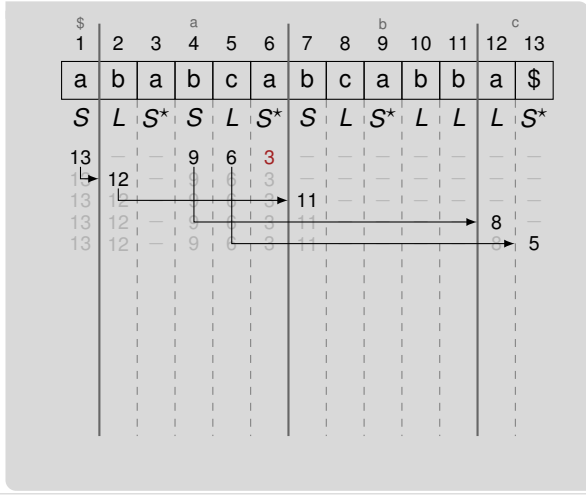

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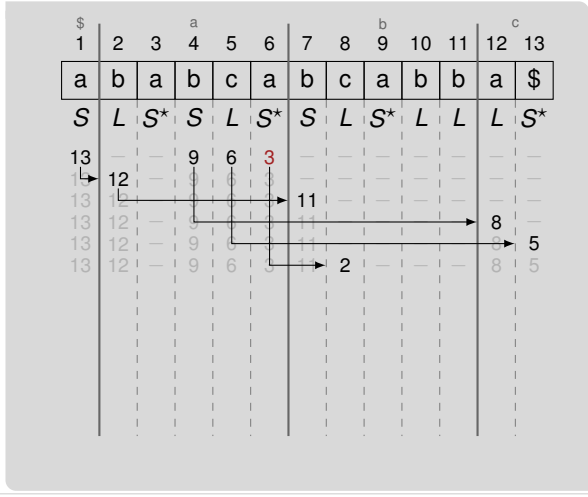

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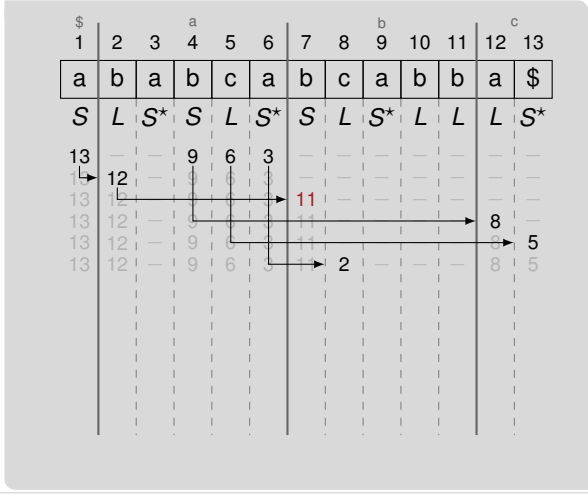

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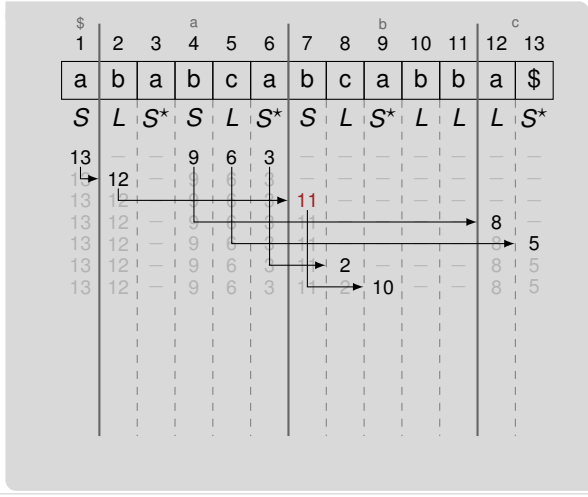

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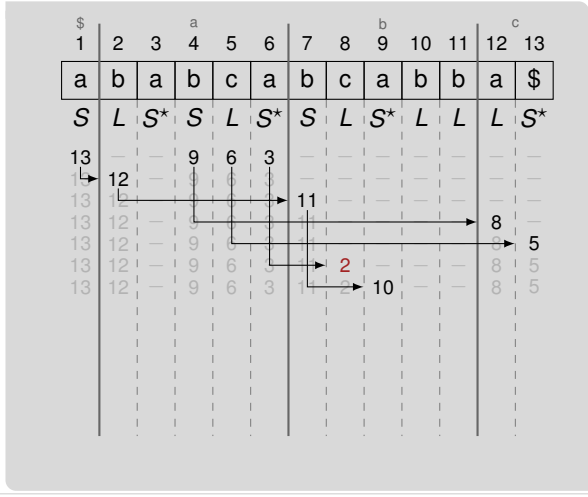

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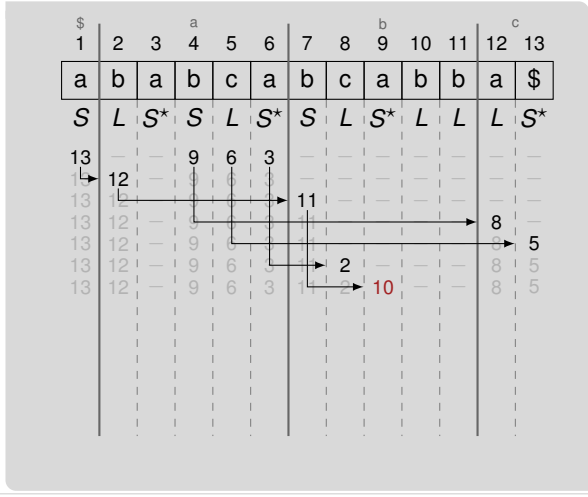

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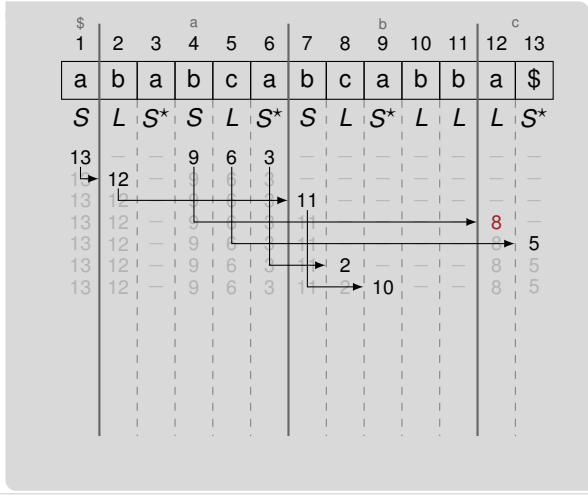

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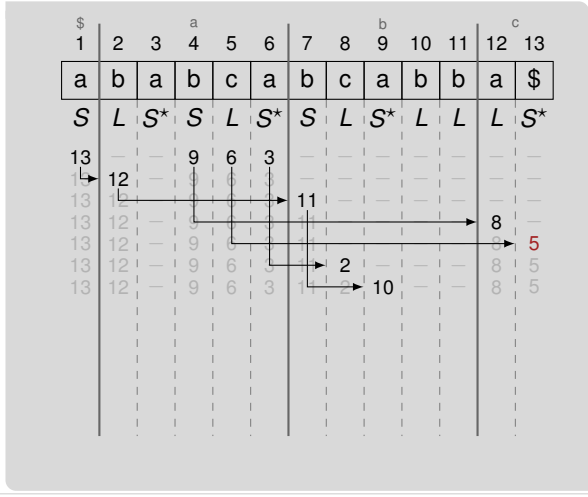

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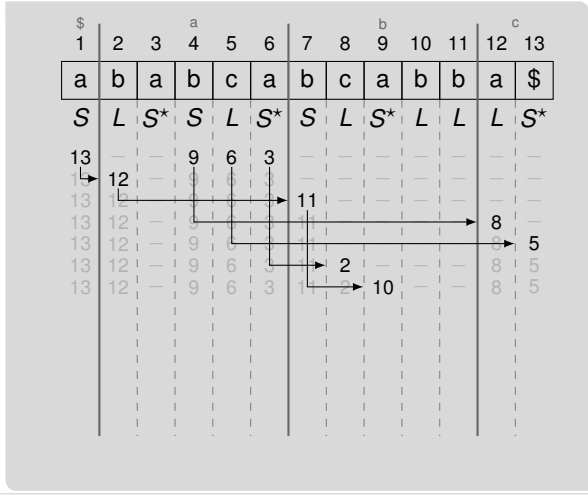

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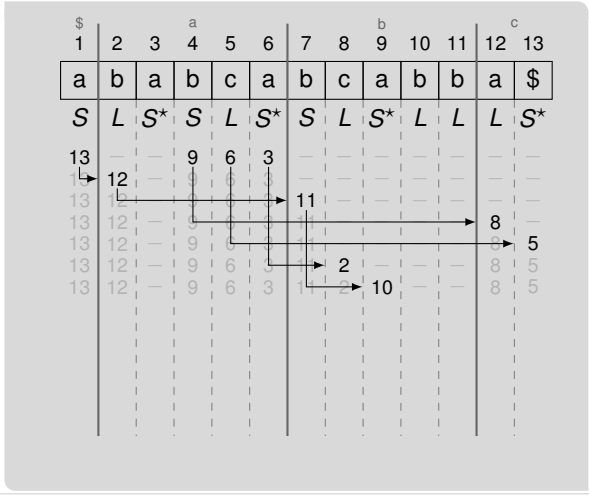

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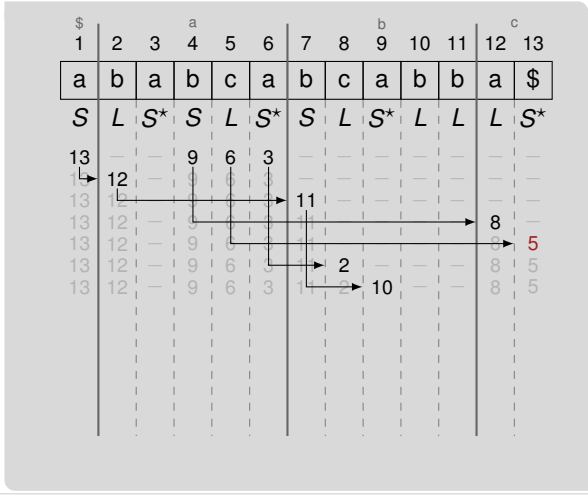

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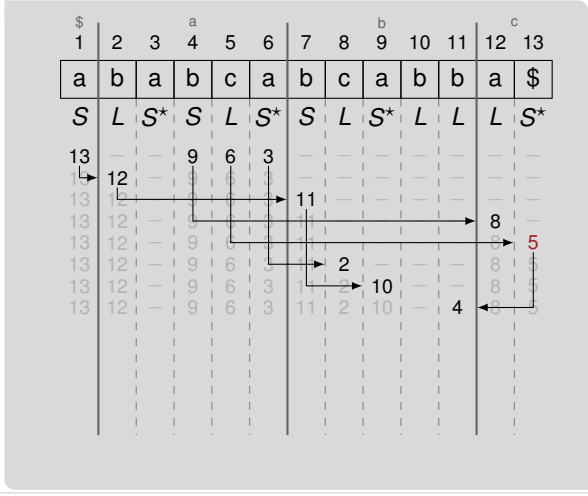

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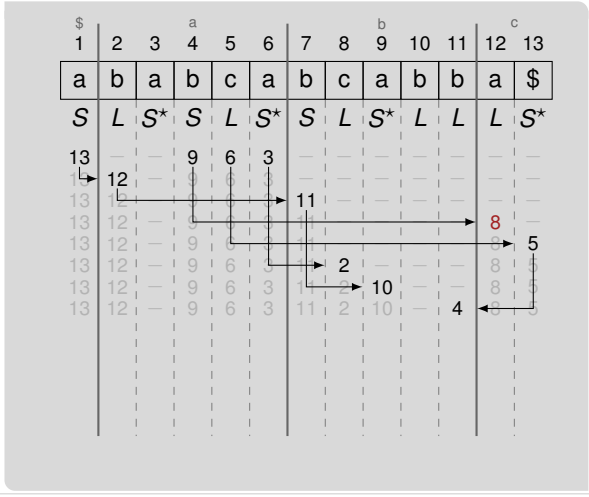

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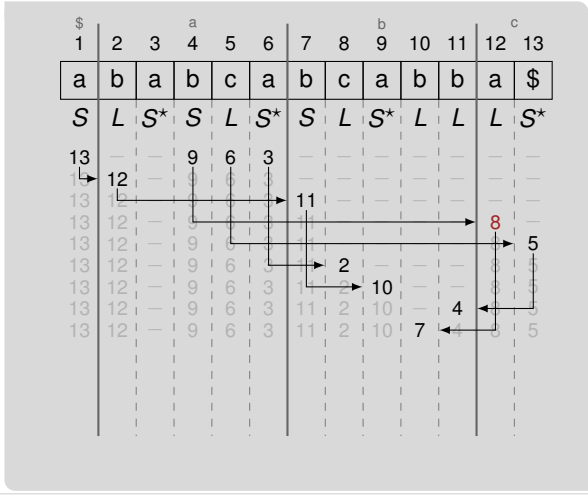

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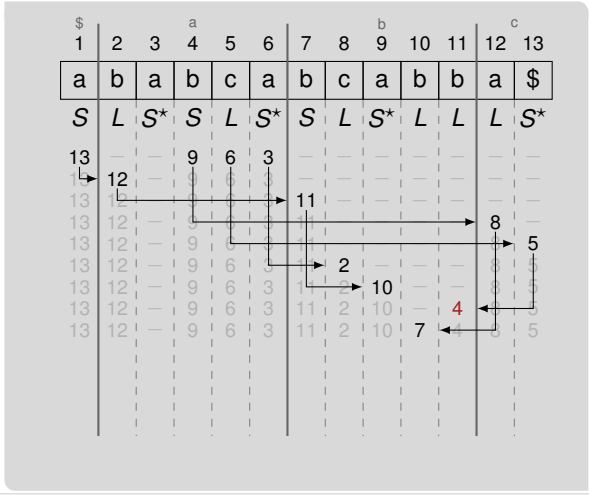

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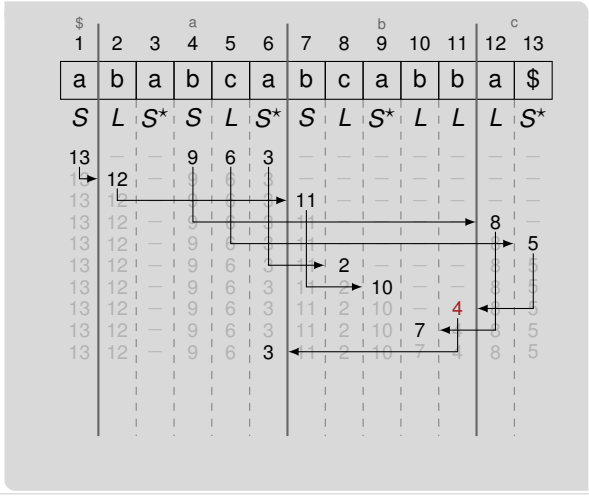

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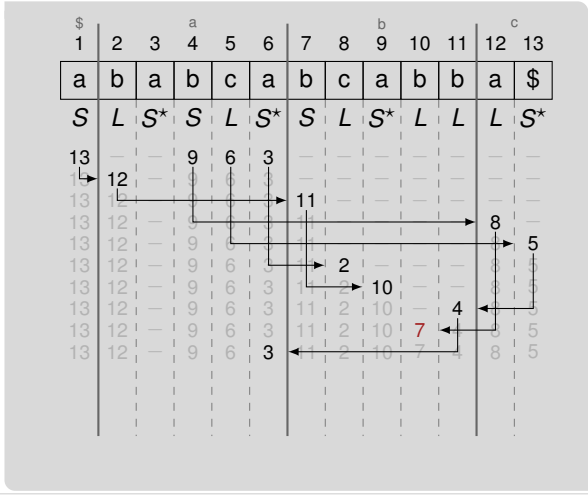

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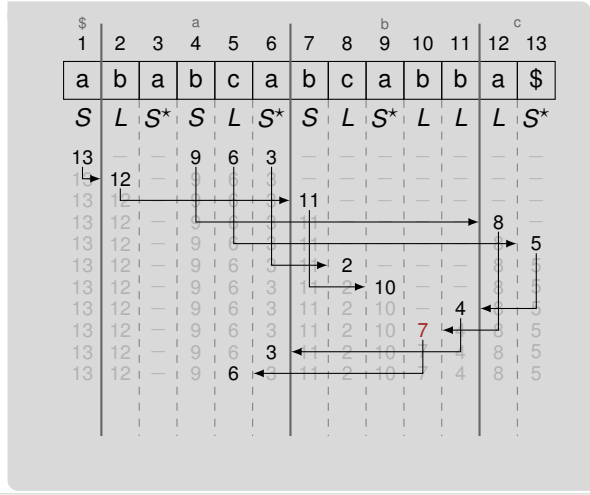

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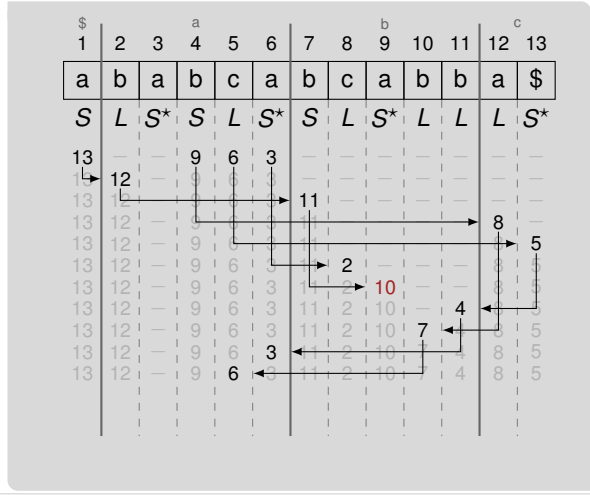

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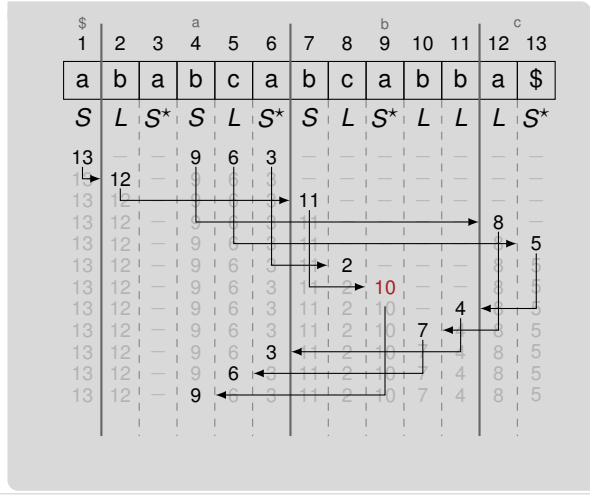

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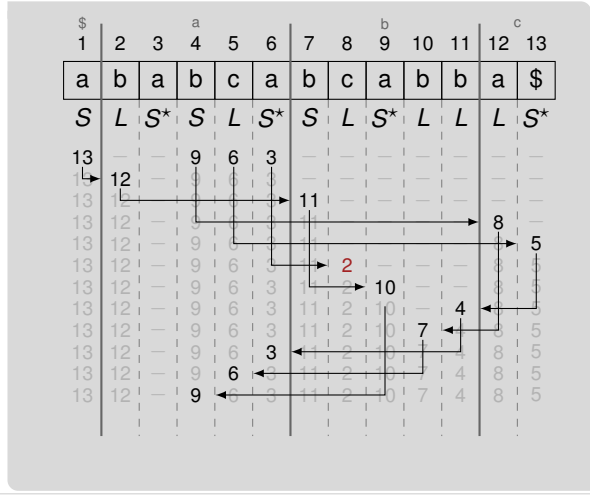

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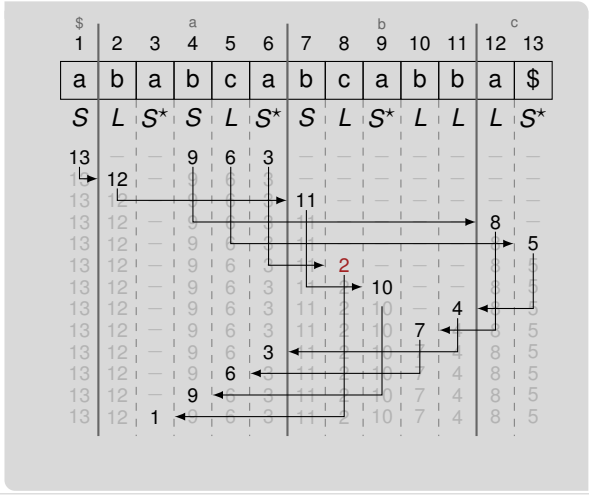

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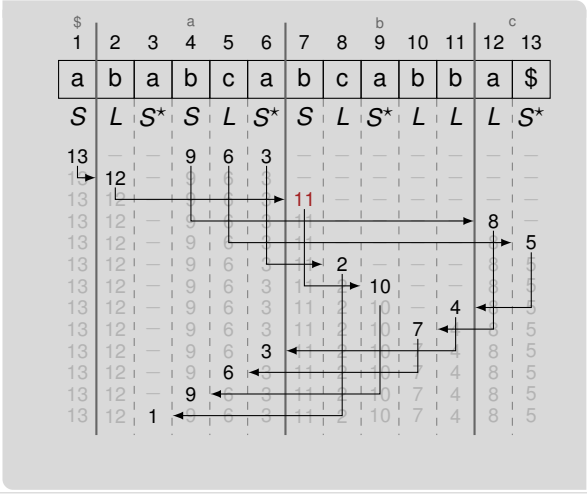

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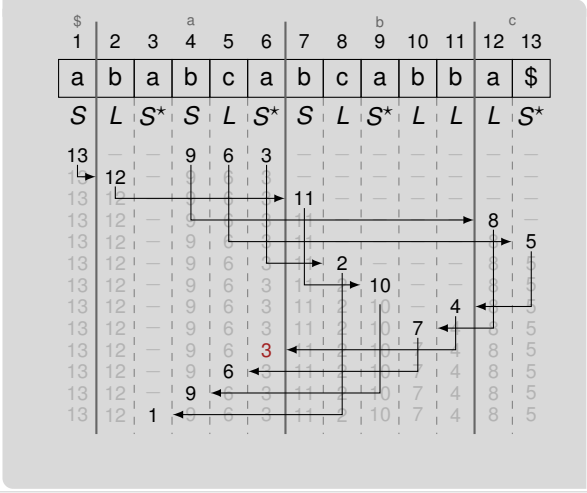

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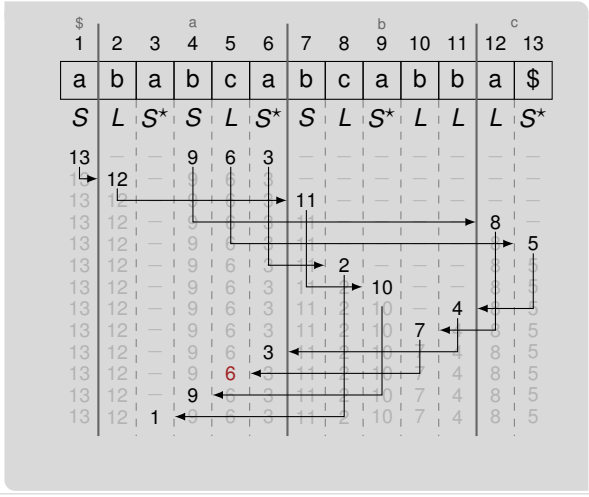

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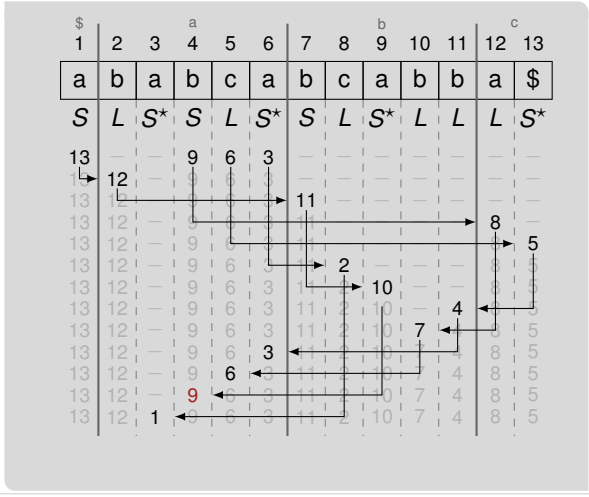

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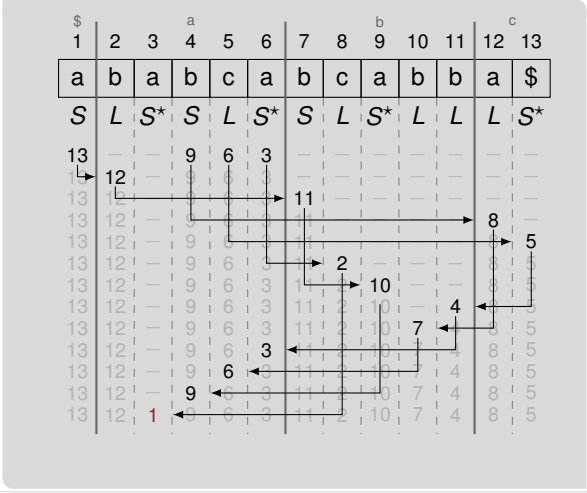

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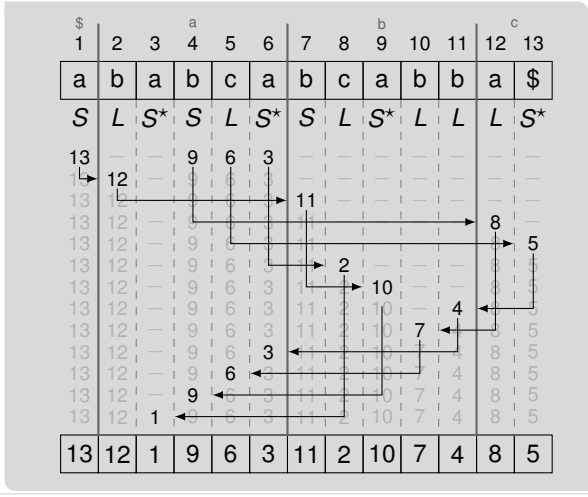

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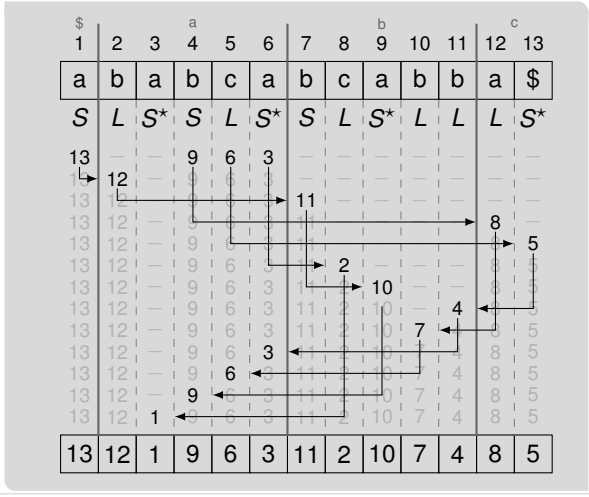

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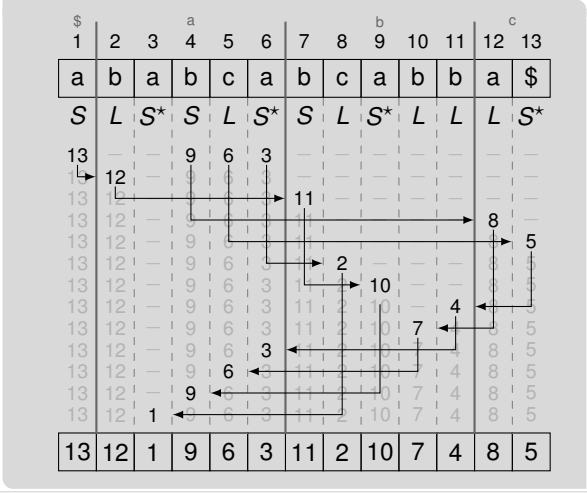
- are all suffixes induced?
- now we only need to sort $S^\star$ suffixes



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Let $i < j$ or $i = j = n$ be text positions, such that $T[j..n]$ is LMS and $\nexists k \in (i, j)$ with $T[k..n]$ is LMS, then we call $T[i..j]$ **LMS-prefix**

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| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *T* | a | b | a | b | c | a | b | c | a | b | b | a | $ |
| *SA* | 13 | 12 | 1 | 9 | 6 | 3 | 11 | 2 | 10 | 7 | 4 | 8 | 5 |
| | $ | a$ | abacabba$ | abba$ | abcabba$ | abcabba$ | ba$ | bacabba$ | bba$ | bcabba$ | bcabba$ | cabba$ | cabcabba$ |

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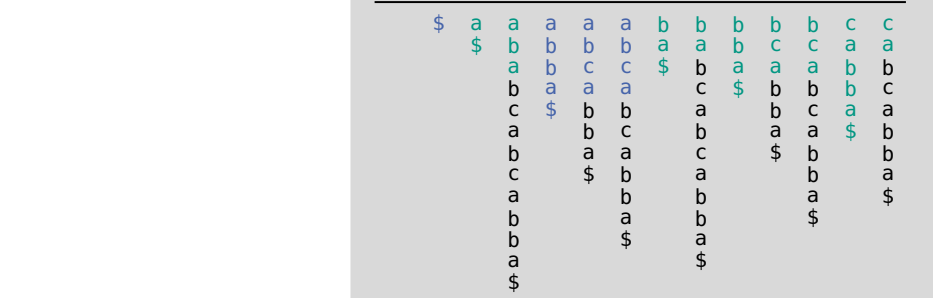

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$
a $
a b a b c a b c a b b a $
a b a b a $
a b c a b b a $
a b c a b c a b b a $
b a $
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$ a$ abacabcabba$ abba$ abcabba$ abcabbcabba$ ba$ babcabcabba$ babca$ bcabcabba$ cabba$ cabcabba$

- $T' = 0122\$$

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## Space Requirements

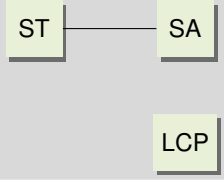
- naive: $O(n \lg n)$ bits
- better: $n\lceil \lg n \rceil + 2\sigma\lceil \lg n \rceil$ bits 

# Conclusion and Outlook

## This Lecture

- suffix trees and suffix arrays
- linear time suffix array construction

## Linear Time Construction



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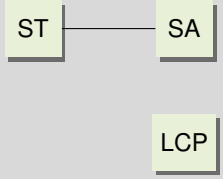

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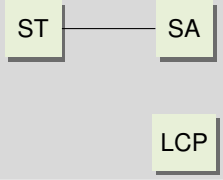


## Next Lecture

- linear time LCP-array construction
- interesting properties of LCP-array
- computing suffix trees using suffix array and LCP-array

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