Algorithms for Memory Hierarchies
Lecture 9

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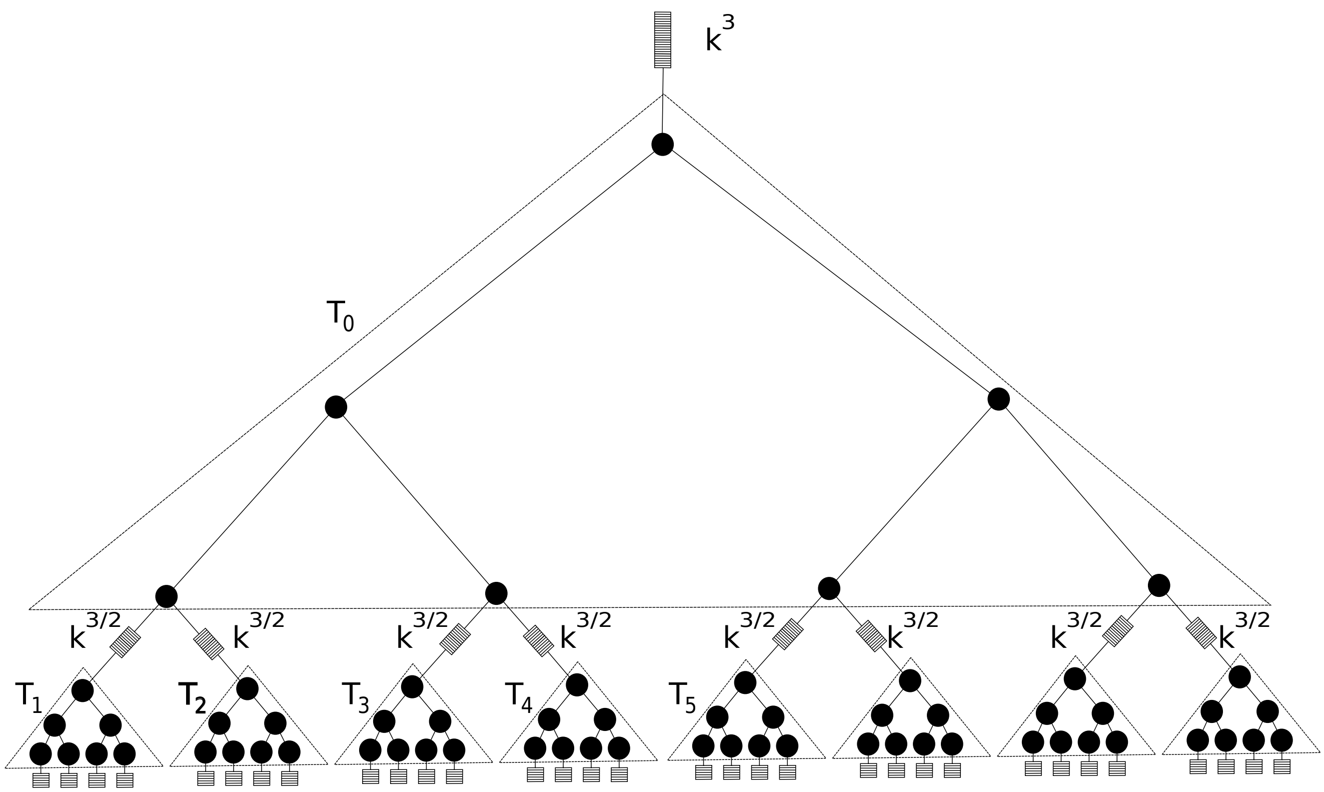
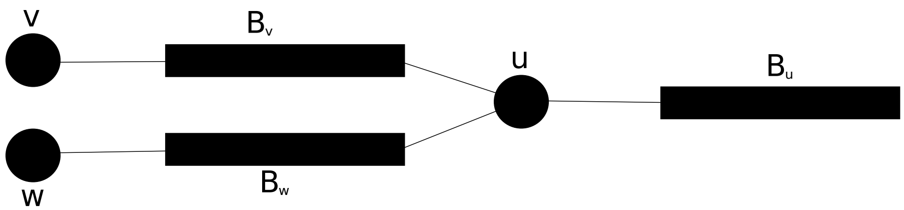
In this lecture we will consider *funnels* – a cache-oblivious data structure that we will use for building funnel heap, a cache-oblivious priority queue.

1 Funnel data structure



Funnels are cache-oblivious data structures used for merging sorted sequences. We can view them as binary trees that have an input buffer at each leaf, and an output buffer at the root. A funnel with k leaves is called a k-funnel. The size of the output buffer of a k-funnel is k^3. A k-funnel has $2k-1$ nodes. We can view a k-funnel as consisting of $\sqrt{k}+1$ $\sqrt{k}$-funnels, one at the top and one for each leaf of the top $\sqrt{k}$-funnel, with output buffers of the bottom funnels acting as the input buffer for the top one. In the example above, T_0 is the top $\sqrt{k}$-funnel, T_1, T_2, ..., $T_{\sqrt{k}}$. The size of a buffer connecting two $\sqrt{k}$-funnels is $\sqrt{k}^3 = k^{\frac{3}{2}}$. Funnels are using the van Emde Boas memory layout, the top $\sqrt{k}$-funnel is stored first, followed by the leaf $\sqrt{k}$-funnels. For our example the memory layout would be: $T_0 T_1 T_2 \ldots T_{\sqrt{k}}$. The $\sqrt{k}$-funnels are also using the van Emde Boas memory layout. The smallest funnel is the 2-funnel and its output buffer size is $2^3 = 8$.

2 Filling the output buffer




```
FILL(u)
while B_u not full do
  | if B_v empty then
  |   | FILL(v);
  | end
  | if B_w empty then
  |   | FILL(w);
  | end
  | perform one merge step;
end
```

Procedure FILL(u)

When filling the buffer B_u we check before each merge step if the buffers B_v and B_w of the two children still contain any elements. If that is not the case, we fill the buffer that is empty.

Theorem 1. *The space complexity of a k-funnel, excluding the input and output buffers, is $O(k^2)$.*

Proof. The space complexity of the k-funnel is defined by the recurrence:

$$\begin{aligned} S(k) &= S(\sqrt{k}) + \sqrt{k}S(\sqrt{k}) + \sqrt{k}k^{\frac{3}{2}} \\ &= (1+\sqrt{k}) \cdot S(\sqrt{k}) + O(k^2) \\ &= O(k^2) \end{aligned} \tag{1}$$

□

Theorem 2. *Assuming $M = \Omega(B^2)$, a merging step using k-funnel that output k^3 elements takes $O(\frac{k^3}{B}\log_{\frac{M}{B}}\frac{k^3}{B})$ I/Os.*

Proof. Let a $\bar{k}$-funnel be the largest funnel that fits in main memory. It will take $O(\bar{k}^2)$ space, with $\bar{k}^2 \leq M$. We shall call it a *base funnel* (see Figure 1).

Consider the path beginning at the root and ending at a leaf of the k-funnel. There will be $\frac{\log k}{\log \bar{k}} = \log_{\bar{k}} k$ base funnels on this path (see Figure 2).

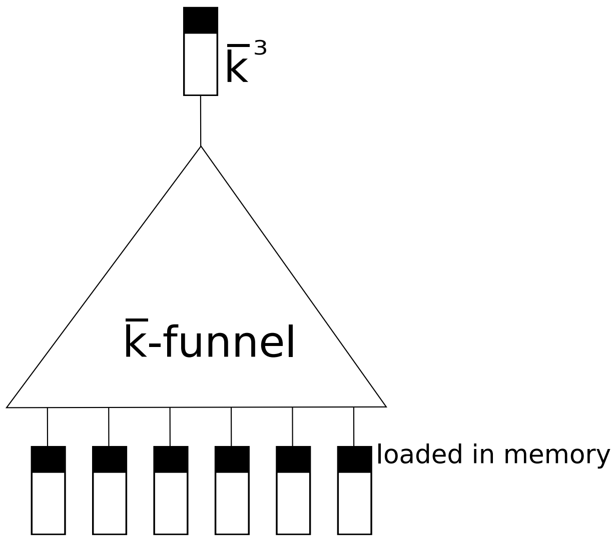


Figure 1: A base funnel – the largest k-funnel that fits in internal memory.

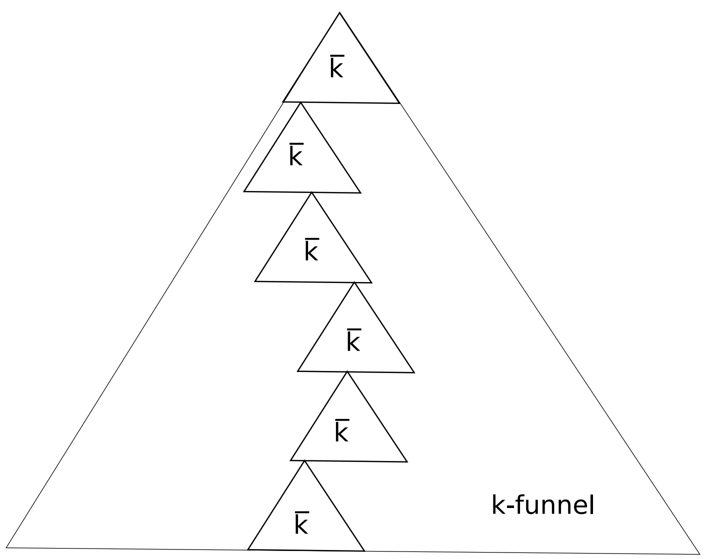


Figure 2: A path within a k-funnel consisting of base funnels.

The I/O complexity of merging $\bar{k}^3$ in a $\bar{k}$-funnel is: $O\left(\bar{k} + \frac{\bar{k}^3}{B}\right)$ I/Os. Since the next biggest funnel is the $\bar{k}^2$-funnel with size $O(\bar{k}^4) > M > B^2$, $\frac{\bar{k}^2}{B} > 1 \Rightarrow \frac{\bar{k}^3}{B} > \bar{k}$. And, therefore, $O\left(\bar{k} + \frac{\bar{k}^3}{B}\right) = O\left(\frac{\bar{k}^3}{B}\right)$

Therefore, we spend $O(\frac{1}{B})$ I/O's per element when operating on base funnels. Each element passes through $\frac{\log k}{\log \bar{k}}$ base funnels. Then the total of I/O's that an element needs to go from an input buffer to an output buffer of a k-funnel is:

$$\begin{aligned} O(\frac{1}{B} \cdot \frac{\log k}{\log \bar{k}}) &= O(\frac{1}{B} \frac{\log k}{\log M}) \\ &= O(\frac{1}{B} \log_M k) \end{aligned}$$

Because we process k^3 elements, the total I/O will be:

$$O\left(\frac{k^3}{B} \log_M k\right) = O\left(\frac{k^3}{B} \log_{M/B} k^3\right) = O\left(\frac{k^3}{B}\left(\log_{M/B} \frac{k^3}{B} + \log_{M/B} B\right)\right)$$

Assuming $M = \Omega(B^2)$, $\log_{M/B} B \leq \log_B B = 1$ and the total I/O complexity of merging using a k-funnel outputting k^3 items from k sorted streams is

$$O\left(\frac{k^3}{B} \log_{\frac{M}{B}} \frac{k^3}{B}\right).$$

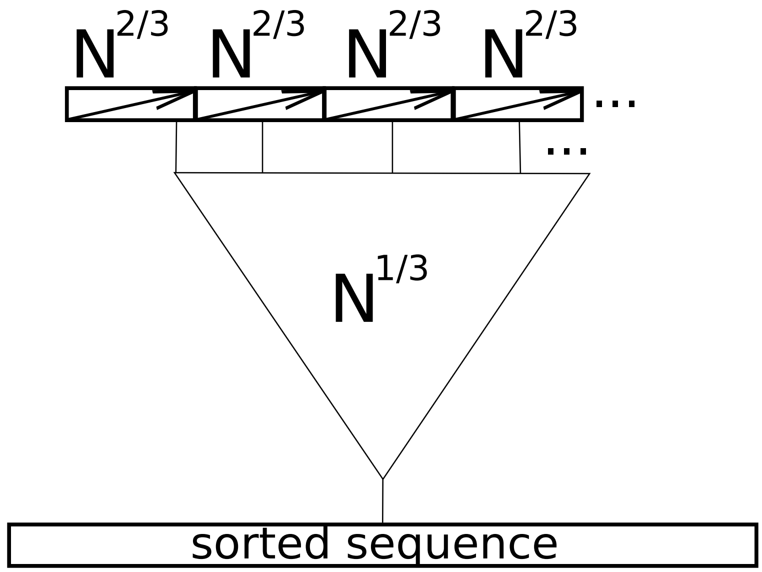
□

3 Funnel sort

Now we show how we can implement a cache-oblivious merge sort using funnels. We cannot simply use an N-funnel because it will take too much space. Instead we will use the following recursive procedure.

1. Split A into $N^{\frac{1}{3}}$ sub-arrays ;
2. Recursively sort each sub-array ;
3. Merge the $N^{\frac{1}{3}}$ streams using a $N^{\frac{1}{3}}$-funnel ;

Procedure Sort(A)



The space complexity of funnel sort is defined by the following recursion

$$S(k) = O(k) + O(N^{\frac{2}{3}}) = O(N)$$

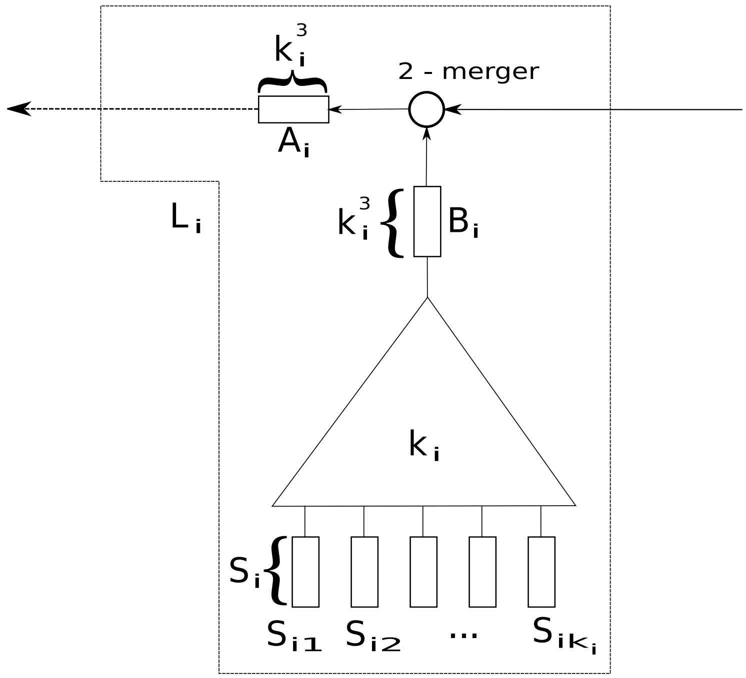
The I/O complexity of funnel sort is:

$$\begin{aligned} Q(N) &= N^{\frac{1}{3}} Q(N^{\frac{2}{3}}) + O(\frac{N}{B} \log_{\frac{M}{B}} \frac{N}{B}) \\ &= O(\frac{N}{B} \log_{\frac{M}{B}} \frac{N}{B}) \end{aligned}$$

which is optimal I/O complexity of sorting N items.

4 Funnel heap

In this section we describe a cache-oblivious priority queue based on the *funnel heap* data structure.



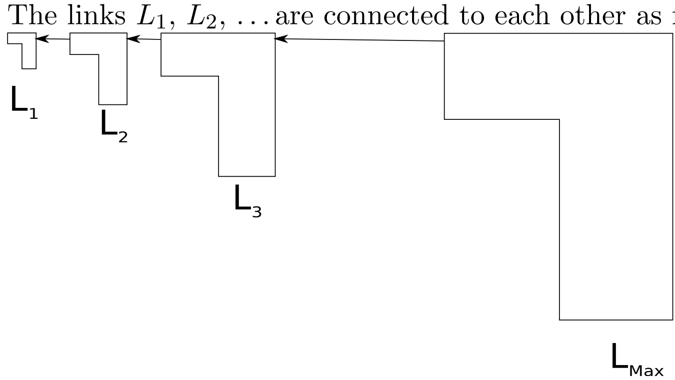
The funnel heap consists of multiple chained links. The image above shows one link L_i. It consists of a k_i-funnel with an output buffer B_i. A 2-merger then merges the elements of the buffer B_i with the elements of the next link and outputs them in the A_i buffer. The size of the A_i and B_i buffers is k_i^3. The size of the k_i input buffers to the k_i-merger is s_i.

The sizes s_i and k_i for each link L_i are defined recursively as follows.

$$(k_1, s_1) = (2, 8)$$
$$(k_i, s_i) = \left(\lceil\lceil \sqrt[3]{s_i} \rceil\rceil, s_{i-1} \cdot (k_{i-1} + 1)\right),$$

where $\lceil\lceil x \rceil\rceil$ represent the smallest number that is a power of 2 and is greater than x.

The links L_1, L_2, ... are connected to each other as follows:



Funnel heap maintains the items in heap order, meaning the items on the path from the first element of buffer A_1 to every leaf s_{ij} are in in non-decreasing order. Thus, the smallest element in the heap is always the element $A_1[0]$. This leads to the following simple procedure for the priority queue DeleteMin() operation. That is we fill the buffer A_1 if it's empty and return the smallest element in it.

Obviously, the hardest part is maintaining the heap order during the insertion.

DeleteMin()
if A_1 *empty* **then**
| Fill(A_1);
end
Return and delete $A_1[0]$;

Insert(x)

1. Let S_{ij} be the first empty leaf buffer;
2. Empty all $L_r (r < i)$ by marking A_i as empty and repeatedly calling DeleteMin();
3. Empty the path from S_{ij} to A_i ;
4. Merge the two sorted sequences producing a single sorted sequence of all the removed items;
5. Place the sorted items on the path from A_1 to S_{ij} in such a way that the buffers $A_r (r \le i)$ and B_i contain the same number of items as before they were removed from it;

Note that the removed elements will fit in the buffers on the path because the total items remaining after all buffers $A_r (r \le i)$ and B_i have been filled is at most $\sum_{r=1}^{i-1} s_r \cdot (k_r + 1)$, which is at most $|s_{ij}| = s_i$ by definition of s_i.

In order to analyse the I/O complexity observe that an item might participate in the removal (and merging) multiple times. However, each time it is removed, it will never be placed in a funnel of lower links than where it was removed from. And once it reaches the largest link L_i (with largest i) in its lifetime, it will move in the future only upward and to the left. The number of I/Os it'll spend going up the link L_i is at most $O\left(\frac{1}{B} \cdot s_i\right)$.

Thus, the total I/Os that we spend on moving the item up within the funnels is

$$\begin{aligned}
\sum_{r=1}^{i} O\left(\frac{1}{B}\log_M s_r\right) &= O\left(\sum_{r=1}^{i} \frac{1}{B}\log_M 2^{(4/3)^r}\right) = O\left(\sum_{r=1}^{i} \frac{1}{B}(4/3)^r \cdot \log_M 2\right) \\
&= O\left(\frac{1}{B}(4/3)^{i_{Max}} \cdot \log_M 2\right) = O\left(\frac{1}{B} \cdot \log_M 2^{(4/3)^{i_{Max}}}\right) \\
&= O\left(\frac{1}{B} \cdot \log_M (s_{Max})\right) = O\left(\frac{1}{B} \cdot \log_M N\right) \\
&= O\left(\frac{1}{B} \cdot \log_{M/B} \frac{N}{B}\right)
\end{aligned}$$

The last equality is under our usual assumption that $M = \Omega(B^2)$.

Thus, the amortized I/O complexity of DeleteMin() and Insert(x) operations on the cache-oblivious priority queue is $O\left(\frac{1}{B} \cdot \log_{M/B} \frac{N}{B}\right)$, just like in the EM model.

5 Applications

Using the cache-oblivious priority queue we can implement all the graph algorithms we have considered in the EM model.